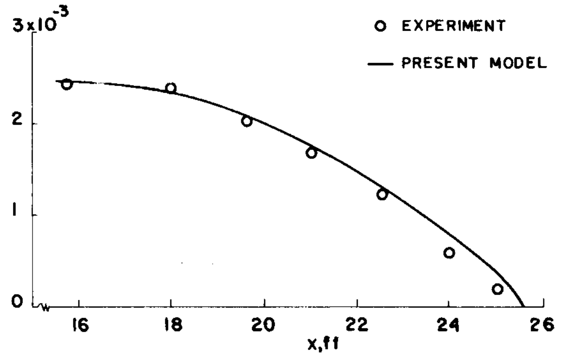

Fig. 4 Prediction of separating flow of Ref. 7.

is, by taking $\tau_\varepsilon = 0.016\ u_e \delta^* \rho(\partial u/\partial y)$. Lower values of the shear in the middle of the boundary layer are predicted by use of the energy equation.

A final result shown in Fig. 4 is the theoretical prediction of the separating flow of Schubauer and Klebanoff. For this calculation the integral energy equation, Eq. (2), has been coupled with the integral, two-strip, boundary-layer method of Baronti,[9] which requires specification of the turbulent shear at $\eta_1 = 0.5$. The present results appear to be at least as accurate as those obtained by Bradshaw et al. by a finite difference method.

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A Vortex Entrainment Model Applied to Slender Delta Wings

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Nomenclature

A = wing aspect ratio
a = wing semispan at chordwise station x
b = wing semispan at trailing edge
C_L = total lift coefficient, $L/\frac{1}{2}\rho U^2 S$
C_N = normal force coefficient, $N/\frac{1}{2}\rho U^2 S$
c = wing chord
K = vortex strength
K' = nondimensional vortex strength, $K/U_c a$
M = sink strength
M' = nondimensional sink strength, $M/U_c a$
N = normal force exerted on wing
S = total wing area
U = freestream velocity
U_c = velocity in cross-flow plane, $U \sin \alpha$
V = resultant velocity along vortex core axis
V' = nondimensional velocity along vortex axis, V/U_c
V_x', V_y', V_z' = nondimensional components of velocity along vortex axis
x, y, z = body axis coordinate system, Fig. 1
Z = complex variable in cross-flow plane, $z + iy$
α = angle of attack
$\gamma(x)$ = strength per unit width of bound vortex
$\Gamma(y)$ = circulation about spanwise station y
δ = angle of inclination of vortex core above delta wing
ρ = density of fluid
ζ = complex variable in nondimensional transformed plane, $(Z^2/a^2 + 1)^{1/2}$

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Introduction

CURRENT experimental studies on bodies of revolution at high angles of attack have shown that the forces and moments developed are greatly affected by the formation of rolled up vortex cores above the lee surface.[1] Thus it is felt that an accurate model of the vortex would aid in the design of fuselage forebodies or slender bodies in general.

Due to geometric simplicity several mathematical models of the vortex flow over a slender, sharp-edged, delta wing have been formulated.[2–5] However, these models generally ignore the entrainment effect of the vortex core, and are found to yield results which are not in agreement with experiment, thus, their extension to the more general case would be of little value.

The technique of Polhamus,[6] referred to as the leading-edge suction analogy, has been found to yield extremely accurate results when compared with experiment. However, there is at present no firm analytical foundation for this result.

It is the intent of this Note to propose a mathematical model of the vortex, which incorporates the previously ignored entrainment effect, and leads to an expression similar to the leading-edge suction analogy.

Mathematical Model

Consider a slender, uncambered, sharp-edged, delta wing as shown in Fig. 1. Since vorticity is shed along the entire leading edge, the vortex core is modeled as a line vortex of strength $K(x)$. The entrainment of mass by the vortex core is accomplished by superimposing a distribution of sinks along the core axis of strength $M(x)$. After being entrained, the mass proceeds along the core axis with velocity V.

The condition that the line vortex remain stationary is accomplished by requiring that the resultant velocity at the axis of the core be directed along the core axis, as indicated in Fig. 1.

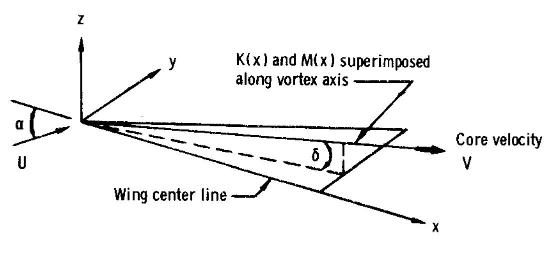

Fig. 1 Illustration of proposed model (half wing shown).

From symmetry, the bound vorticity at the wing centerline must be perpendicular to the centerline. Additionally, in order to form a rolled-up vortex core, the bound vorticity must leave at the leading edge. Thus, it is assumed that the bound vortex lines are in the spanwise direction.

Analysis

Using slender body theory, the problem reduces to a flat plate of width $2a$ placed normal to a freestream of velocity U_c

$$U_c = U \sin\alpha \tag{1}$$

Behind the plate there are two vorticies of strength $-K$ and K, superimposed on sinks of strength M, and located at the points

$$Z_1 = z_0 + iy_0, \qquad Z_2 = z_0 - iy_0 \tag{2}$$

The boundary conditions in this plane are: i) flow tangent to the plate, ii) smooth outflow at the plate edges, and iii) stationary vortex-sink core.

From the assumption of slender body and conical flow, the strengths of the vortices and sinks are found to increase linearly in the downstream direction, while the velocity along the core axis remains constant.

By introduction of the nondimensional transformed plane

$$\zeta = (Z^2/a^2 + 1)^{1/2} \tag{3}$$

Boundary condition (i), tangent flow, is automatically satisfied. Proceeding as in the analysis of Ref. 7, and introducing nondimensional variables M', K', and V', it is found that boundary condition (ii), smooth outflow, requires

$$M'(\zeta_1+\zeta_2)/\zeta_1\zeta_2 + iK'(\zeta_1-\zeta_2)/\zeta_1\zeta_2 = -1 \tag{4}$$

additionally, boundary condition (iii), stationary vortex-sink, requires

$$\frac{(M'+iK')Z_1}{(\zeta_1-\zeta_2)\zeta_1 a} + \frac{(M'-iK')a}{2Z_1\zeta_1{}^2} - \frac{Z_1}{a\zeta_1} = -V_z' + iV_y' \tag{5}$$

Equation (5) is a complex equation which can be separated into real and imaginary components.

Substitution of the vortex core locations, obtained from Ref. 8, allows the simultaneous solution of Eq. (4) and the real and imaginary portion of Eq. (5), for the corresponding values of M', K', and V'.

The total lift developed by the delta wing is due to the contributions of the free vortex system and the bound vortices.

Consider the free vortex system as illustrated in Fig. 2. Using the standard momentum analysis, the normal force per unit chord is determined as

$$dN(x) = 4\pi M(x)\rho(U_c - V\sin\delta)\,dx \tag{6}$$

Introducing nondimensional variables, M', K', V', C_N, and carrying out the integration, yields

$$C_N = 4\pi M'(1 - V'\sin\delta)\sin^2\alpha \tag{7}$$

Thus, the contribution of the lift due to the free vortices becomes

$$C_{L_f} = 4\pi M'(1 - V'\sin\delta)\cos\alpha\sin^2\alpha \tag{8}$$

It is next desired to determine the component of the lift due to the bound vorticity, the assumed distribution of which is indicated in Fig. 3. The normal force of these bound vortices is determined by applying the Kutta-Joukowski theorem, using the velocity component parallel to the wing chord. The normal force per unit span is determined as

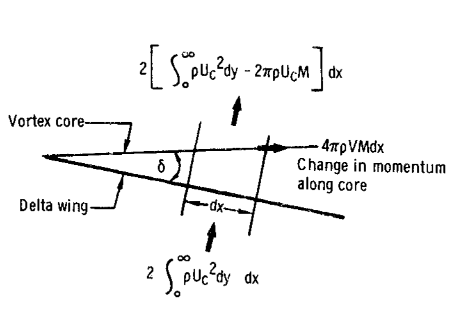

Fig. 2 Momentum consideration of free vorticity (side view of delta wing).

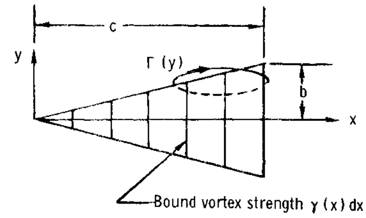

Fig. 3 Plan view of delta wing and bound vortex system.

$$dN(y) = \rho U\cos\alpha\Gamma(y) \tag{9}$$

and thus the normal force contribution is given by

$$N = 2\rho U\cos\alpha\int_0^b \Gamma(y)\,dy \tag{10}$$

where $\Gamma(y)$ is determined from

$$\Gamma(y) = 2\pi\int_{x(y)}^c \gamma(x)\,dx \tag{11}$$

Because all of the bound vortices of strength $\gamma(x)dx$ form the concentrated core of strength $K(x)$, it is possible to relate their strengths. Using the geometric relationship

$$a(x) = (A/4)x \tag{12}$$

and equating vortex strengths yields

$$\gamma(x) = K'(A/4)U\sin\alpha \tag{13}$$

Substitution of Eq. (13) into Eqs. (11) and (10), and carrying out the integration yields upon introduction of nondimensional coefficients

$$C_N = \pi K'A\cos\alpha\sin\alpha \tag{14}$$

Thus the contribution of the bound vortices to the lift is given as

$$C_{L_B} = \pi K'A\cos^2\alpha\sin\alpha \tag{15}$$

and hence the total lift coefficient is given as

$$C_L = \pi K'A\cos^2\alpha\sin\alpha + 4\pi M'(1 - V'\sin\delta)\cos\alpha\sin^2\alpha \tag{16}$$

The leading-edge suction analogy of Polhamus,[6] yields

$$C_L = Kp\cos^2\alpha\sin\alpha + Kv\cos\alpha\sin^2\alpha \tag{17}$$

Hence, the present mathematical model leads to an expression similar to the leading-edge suction analogy.

Utilizing the previously indicated method to solve for K', M', and V' thus determines the lift coefficient. This is presented in Fig. 4, along with experimental data obtained from Refs. 9 and 10. The value of the lift coefficient as determined by Polhamus is presented for comparison.

Conclusion

The proposed mathematical model is seen to yield results which, although not as accurate as the leading-edge suction

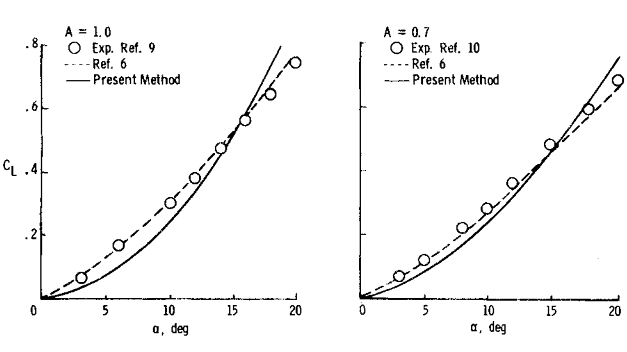

Fig. 4 Comparison of present theory for lift delta wings.

analogy, are found to be in agreement with the experiment. Although the present technique requires experimental data, in the form of the vortex core locations, the model does account for the previously ignored mass entrainment of the vortex core.

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Determination of Bending Influence Coefficients from Bending Slope Data

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BENDING influence coefficients for a cantilever beam are defined as follows: 1) $H_h(x, \bar{x})$ = vertical deflection at "x" due to unit force at "$\bar{x}$" (in./lb); 2) $H_\theta(x, \bar{x})$ = vertical deflection at "x" due to unit moment at "$\bar{x}$" (in./in. lb); 3) $\theta_h(x, \bar{x})$ = angular deflection at "x" due to unit force at "$\bar{x}$" (rad/lb); 4) $\theta_\theta(x, \bar{x})$ = angular deflection at "x" due to unit moment at "$\bar{x}$" (rad/in. lb). Direct determination of these quantities would normally require several sets of measurements for each separate loading. It will be shown that all of these quantities can be found either directly or by simple quadrature from a single set of measurements, namely, the angular deflection distribution due to a unit moment applied at the tip.

The analytical expressions for the various influence coefficients in terms of bending stiffness, EI, are

$$\begin{cases} H_h(x, \bar{x}) = \displaystyle\int_0^x \frac{(x-\xi)(\bar{x}-\xi)}{EI(\xi)} d\xi, & x \leqq \bar{x} \\ H_h(x, \bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{(x-\xi)(\bar{x}-\xi)}{EI(\xi)} d\xi, & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} H_\theta(x, \bar{x}) = \displaystyle\int_0^x \frac{(x-\xi)}{EI(\xi)} d\xi, & x \leqq \bar{x} \\ H_\theta(x, \bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{(x-\xi)}{EI(\xi)} d\xi, & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_h(x, \bar{x}) = \displaystyle\int_0^x \frac{(\bar{x}-\xi)}{EI(\xi)} d\xi, & x \leqq \bar{x} \\ \theta_h(x, \bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{(\bar{x}-\xi)}{EI(\xi)} d\xi, & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_\theta(x, \bar{x}) = \displaystyle\int_0^x \frac{d\xi}{EI(\xi)}, & x \leqq \bar{x} \\ \theta_\theta(x, \bar{x}) = \displaystyle\int_0^{\bar{x}} \frac{d\xi}{EI(\xi)}, & x \geqq \bar{x} \end{cases}$$

From these equations it is seen that

$$H_h(x, \bar{x}) = H_h(\bar{x}, x)$$
$$H_\theta(x, \bar{x}) = \theta_h(\bar{x}, x)$$
$$\theta_\theta(x, \bar{x}) = \theta_\theta(\bar{x}, x)$$

If $\phi(x)$ is the angular deflection at x due to a unit moment at the tip, then

$$\phi(x) = \int_0^x \frac{d\xi}{EI(\xi)}$$

Hence

$$\frac{1}{EI(x)} = \frac{d\phi}{dx}$$

Inserting the expression for $(1/EI)$ in the equations for the influence coefficients and simplifying, we obtain

$$\begin{cases} H_h(x, \bar{x}) = (x+\bar{x}) \displaystyle\int_0^x \phi(\xi)\, d\xi - 2 \int_0^x \phi(\xi)\xi\, d\xi & x \leqq \bar{x} \\ H_h(x, \bar{x}) = (x+\bar{x}) \displaystyle\int_0^{\bar{x}} \phi(\xi)\, d\xi - 2 \int_0^{\bar{x}} \phi(\xi)\xi\, d\xi & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} H_\theta(x, \bar{x}) = \displaystyle\int_0^x \phi(\xi)\, d\xi & x \leqq \bar{x} \\ H_\theta(x, \bar{x}) = (x-\bar{x})\phi(\bar{x}) + \displaystyle\int_0^{\bar{x}} \phi(\xi)\, d\xi & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_h(x, \bar{x}) = (\bar{x}-x)\phi(x) + \displaystyle\int_0^x \phi(\xi)\, d\xi & x \leqq \bar{x} \\ \theta_h(x, \bar{x}) = \displaystyle\int_0^{\bar{x}} \phi(\xi)\, d\xi & x \geqq \bar{x} \end{cases}$$

$$\begin{cases} \theta_\theta(x, \bar{x}) = \phi(x) & x \leqq \bar{x} \\ \theta_\theta(x, \bar{x}) = \phi(\bar{x}) & x \geqq \bar{x} \end{cases}$$

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Index category: Aircraft Vibration.

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Stiffness of Orthotropic Materials and Laminated Fiber-Reinforced Composites

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Introduction

CHU[1] considered a laminated fiber-reinforced composite with 40% of the fibers aligned with the 1-axis in Fig. 1 and the remaining 60% at ±45° to the 1-axis. His description of the laminate unfortunately did not include the number of layers and the stacking sequence. He was perplexed by the observation that the Young's modulus in an x-direction (E_x) other than the 1-direction was larger than in the 1-direction (E_1).† He concluded

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Index categories: Properties of Materials; Structural Composite Materials (Including Coatings); Structural Static Analysis.

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† See Ref. 2 for notation.